

CONGRUENCES OF LINES IN \mathbb{P}^5 , QUADRATIC NORMALITY, AND COMPLETELY EXCEPTIONAL MONGE-AMPÈRE EQUATIONS

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ABSTRACT. The existence is proved of two new families of locally Cohen-Macaulay sextic threefolds in \mathbb{P}^5 , which are not quadratically normal. These threefolds arise naturally in the realm of first order congruences of lines as focal loci and in the study of the completely exceptional Monge-Ampère equations. One of these families comes from a smooth congruence of multidegree $(1, 3, 3)$ which is a smooth Fano fourfold of index two and genus 9.

INTRODUCTION

In 1901 Francesco Severi [Sev01] proved his celebrated theorem saying that the unique surfaces in \mathbb{P}^5 whose secant variety is strictly contained in \mathbb{P}^5 are the Veronese surface and the cones. It follows that if X is a nondegenerate surface in \mathbb{P}^4 , then it is *linearly normal* (i.e. $h^1(\mathcal{I}_X(1)) = 0$) unless it is the (projected) Veronese surface. Note that a projection of a non-degenerate cone fails to be isomorphic in the vertex, so cones in \mathbb{P}^4 are also linearly normal. As an easy corollary, it follows that if X is a nondegenerate threefold in \mathbb{P}^5 , then it is linearly normal.

An interesting feature of the projected Veronese surface V is that its trisecant lines form a *linear congruence of lines* in \mathbb{P}^4 , i.e. a 3-dimensional linear section of $\mathbb{G}(1, 4)$, the Grassmannian of lines in \mathbb{P}^4 . This allows to characterize V as the degeneracy locus of a general bundle morphism $\phi: \mathcal{O}_{\mathbb{P}^4}^{\oplus 3} \rightarrow \Omega_{\mathbb{P}^4}(2)$ ([Ott92]). It gives also the following Ω -resolution for the ideal of V :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 3} \rightarrow \Omega_{\mathbb{P}^4}(1) \rightarrow \mathcal{I}_V(2) \rightarrow 0,$$

which immediately implies that none of the quadrics containing the hyperplane section of V lifts to a quadric containing V ([Cha89]).

In \mathbb{P}^5 an analogous construction brings to the definition of the *Palatini scroll*, a smooth threefold X defined as degeneracy locus of a general morphism $\mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \rightarrow \Omega_{\mathbb{P}^5}(2)$. X results to be *not quadratically normal*, i.e. $h^1(\mathcal{I}_X(2)) \neq 0$.

These analogies motivated the following question, posed by Peskine and Van de Ven, on the non necessarily smooth threefolds in \mathbb{P}^5 :

Problem 1. *Is the Palatini scroll the unique threefold in \mathbb{P}^5 which is not quadratically normal? If it is not the unique example, classify such threefolds (see [Sch90], where the question is ascribed to Peskine).*

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More generally, one can ask to classify the codimension two subvarieties X in \mathbb{P}^n , which are not $(n-3)$ -normal (i.e. $h^1(\mathcal{I}_X(n-3)) \neq 0$).

Subsequently, F. L. Zak in [Zak97] has generalized Problem 1, posing the following conjecture about j -normality of non-degenerate projective m -dimensional subvarieties in \mathbb{P}^n .

Conjecture. *Let i and j be two integers such that $i \geq 1$ and $j \geq 0$; then*

- (1) $H^i(\mathbb{P}^n, \mathcal{I}_X(j)) = 0$ for $i + j < \frac{m}{n-m-1}$;
- (2) for $i + j = \frac{m}{n-m-1}$ it is possible to “classify” all the varieties for which $H^i(\mathbb{P}^n, \mathcal{I}_X(j)) \neq 0$.

Zak also suggested that examples on the boundary of the conjecture could be constructed as focal loci of some *first order congruences* in \mathbb{P}^n (a congruence is a flat family of lines of dimension $n-1$ and its order is the number of its lines passing through a general point of the space; for generalities on congruences, see [DP04]). As we have seen in the case of \mathbb{P}^4 , all the known examples are varieties whose l -secant lines (with l appropriate integer) give linear congruences, which are special examples of congruences of order one. In the Chow ring of $\mathbb{G}(1, n)$, every congruence is a linear combination with integer coefficients of the Schubert cycles of dimension $n-1$. The sequence of the coefficients (a_0, a_1, \dots, a_k) , $k = \lfloor \frac{n-1}{2} \rfloor$, is called the multidegree of the congruence, the first integer a_0 is precisely its order. For instance, the multidegree of a linear congruence B in \mathbb{P}^5 is $(1, 3, 2)$, this means precisely that the lines of B contained in a general hyperplane fill a hypersurface of degree 3, and that the number of lines contained in a general \mathbb{P}^3 is 2. Since B is formed precisely by the 4-secant lines of its focal locus X , a Palatini threefold in \mathbb{P}^5 , we find in this way that X cannot be contained in a cubic hypersurface while, conversely, its hyperplane section is.

This remark shows the connections with the *lifting problem*: given an integral ν -dimensional variety $X \subset \mathbb{P}^n$, it concerns finding conditions on d, ν, n and σ so that any degree σ hypersurface in \mathbb{P}^{n-1} containing the hyperplane section $S = X \cap H$ of X (H a general hyperplane) lifts to a hypersurface of degree σ containing X . In view of the cohomology of the exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_X(k) \xrightarrow{\cdot H} \mathcal{I}_X(k+1) \rightarrow \mathcal{I}_S(k+1) \rightarrow 0,$$

it is clear that, if X is k -normal, then all hypersurfaces of degree $k+1$ containing S lift to hypersurfaces of the same degree containing X (but not vice-versa). Instead, if $k+1$ is a *non-lifting level* for X (i.e. the restriction map $H^0(\mathcal{I}_X(k+1)) \rightarrow H^0(\mathcal{I}_S(k+1))$ is not onto for a general hyperplane H), then X is necessarily non k -normal.

The main results about the lifting problem have been obtained in the case of subvarieties of codimension two. Laudal [Lau78], Gruson-Peskine [GP82], Strano [Str87] solved the problem in the case of curves in \mathbb{P}^3 , whereas the case of higher dimensional varieties was studied, among others, by Mezzetti-Raspanti [MR90], Mezzetti [Mez92], [Mez94], Chiantini-Ciliberto [CC93], Roggero [Rog01], [Rog03], [Ro03a].

The result for threefolds in \mathbb{P}^5 in the case $\sigma = 3$ is the following ([Mez94], [Ro03a]):

Theorem 1. *If X is a locally Cohen-Macaulay (LCM for short in the following) integral threefold in \mathbb{P}^5 , if $\deg(X) > 7$ and $h^0(\mathcal{I}_{X \cap H}(3)) \neq 0$, then $h^0(\mathcal{I}_X(3)) \neq 0$. Moreover the only LCM threefolds of degree 7 with $\sigma = 3$ for which the assertion is not true are the Palatini threefold and its degenerations.*

Our purpose here is to complete this result finding all the LCM threefolds with $h^0(\mathcal{I}_{X \cap H}(3)) \neq 0$ and $h^0(\mathcal{I}_X(3)) = 0$.

We prove:

Theorem 2. *Let X be a non-degenerate LCM integral threefold of \mathbb{P}^5 of degree $d \leq 6$ and (arithmetic) sectional genus π . Then:*

- (1) *if either $d \leq 5$ or $d = 6$ and $\pi \neq 1, 2$, X is contained in a quadric or in a cubic and is 2-normal;*
- (2) *if $d = 6$ and $\pi = 2$, X is contained in a quadric or in a cubic and $h^1(\mathcal{I}_X(2)) \leq 1$.*

Theorem 3. *There exist two families of LCM threefolds, non singular in codimension one of degree 6 in \mathbb{P}^5 with sectional genus one, such that $h^1(\mathcal{I}_X(2)) = 1$ and $h^0(\mathcal{I}_{X \cap H}(3)) \neq 0$, but $h^0(\mathcal{I}_X(3)) = 0$.*

Our examples give an answer in the negative to Problem 1, at least if one allows singularities. In fact, both examples are singular and non singular in codimension 1. A threefold of the first family is obtained from a particular case of a linear congruence of lines, the focal locus is reducible in two components, one of them is a *parasitical* \mathbb{P}^3 (see Section 2). The second family comes from the congruences which are associated to the completely exceptional Monge-Ampère systems of differential equations. The existence of this family was suggested by Agafonov-Ferapontov: in their article [AF01] they introduce a surprising construction which allows to associate to a system of PDE of conservation laws in $n - 1$ variables a congruence of lines B in \mathbb{P}^n . If the system is of Temple class (also called *T-system*, see Section 3 for the definition), then B has order one, and the focal locus is a subvariety X in \mathbb{P}^n of codimension two, such that the lines of B through a general point of X form a planar pencil of lines (see [DM05]). If $n \leq 4$, it is proved in [AF01] that a congruence associated to a T-system is always linear. The example we exhibit shows that this is no longer true in \mathbb{P}^5 , indeed we get a congruence of multidegree $(1, 3, 3)$. It results to be an irreducible component of a (special) quadratic congruence, *i.e.* of a subvariety of $G(1, 5)$, which is cut by three hyperplanes and one quadric. This congruence, seen as a subvariety of the Grassmannian, is also interesting from other points of view. It is a new example of a smooth variety of dimension 4 covered by lines, such that the number of lines passing through its general point is 4. It results to be a Fano fourfold of index two and genus 9. This was studied by Mukai in [Mu88], who had given an embedding of it in the Grassmannian of 2-planes in \mathbb{P}^5 . We note that Mukai's embedding is obtained by showing the existence of a "good" vector bundle of rank three on the Fano variety, while our embedding comes from a rank two vector bundle.

We also show that our examples exhaust all the possibilities for integral threefolds coming from congruences of type $(1, 3, a_2)$, although we do not exclude the possibility of having more than two families in Theorem 3.

This article is structured as follows: in Section 1, we prove Theorem 2 about threefolds of degree ≤ 6 in \mathbb{P}^5 . In Section 2 we construct the congruences giving rise to the two families of non-2-normal threefolds of degree 6 and sectional genus 1. Finally in Section 3 we explain the connections with the completely exceptional Monge-Ampère equations.

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1. THREEFOLDS OF DEGREE ≤ 6 IN \mathbb{P}^5

In this section X will always denote a locally Cohen-Macaulay (LCM for short) threefold of \mathbb{P}^5 .

If $\deg X \leq 4$, then X is contained in a quadric ([Rog03]). This implies that it is arithmetically Cohen-Macaulay (see [Kwa99], Theorem 2.1). It remains to consider the cases $d = 5, 6$.

The following simple lemma will allow us to prove the 2-normality of the projections of some rational normal scrolls.

Lemma 4. *Let $X \subset \mathbb{P}^n$ be a (nondegenerate) projective variety and $Y \subset X$ a codimension one subvariety. If Y is k -normal and one of the following assumptions is satisfied:*

- (1) *Y is a hyperplane section of X and X is $(k-1)$ -normal,*
- (2) *Y is such that $\deg(Y) > \deg(X)$ and there does not exist any hypersurface V of degree $\leq k$ which contains Y but not X ,*

then X is k -normal too.

Proof. Case (1) follows immediately from the cohomology sequence of

$$0 \rightarrow \mathcal{I}_X(k-1) \xrightarrow{\cdot H} \mathcal{I}_X(k) \rightarrow \mathcal{I}_Y(k) \rightarrow 0.$$

Let us now assume the hypothesis of case (2). Consider the k -tuple embedding $v_k(X)$ of X in \mathbb{P}^N , $N := \binom{n+k}{k} - 1$. Its linear span $\langle v_k(X) \rangle$ is a \mathbb{P}^M , $M \leq N$. If $X \subset \mathbb{P}^n$ is not k -normal, then $v_k(X)$ is not linearly normal, *i.e.* there exist a linear projection $\pi: \mathbb{P}^{M+1} \dashrightarrow \mathbb{P}^M$ and a nondegenerate variety $\tilde{X} \subset \mathbb{P}^{M+1}$ such that $\pi|_{\tilde{X}}$ is an isomorphism onto $v_k(X)$. Therefore, for each subvariety Y of X , there exists a subvariety $\tilde{Y} \subset \tilde{X}$ such that $\pi|_{\tilde{Y}}$ is an isomorphism onto $v_k(Y)$. In particular, if $\dim(\langle \tilde{Y} \rangle) > \dim(\langle v_k(Y) \rangle)$, Y is not k -normal.

We observe that, in general, if $Z \subset \mathbb{P}^n$ is a nondegenerate variety, then $v_k(Z) \subset \mathbb{P}^N$ is degenerate (*i.e.* $M < N$) if and only if there exists a hypersurface of degree k , $V \subset \mathbb{P}^n$, such that $Z \subset V$.

Assumption (2) implies that $\langle v_k(X) \rangle = \langle v_k(Y) \rangle$. If now we suppose that X is not k -normal, there exists a \tilde{Y} as above, and $\deg(\tilde{Y}) = \deg(v_k(Y)) > \deg(v_k(X)) = \deg(\tilde{X})$. Then, either $\langle \tilde{X} \rangle = \langle \tilde{Y} \rangle$ and $\langle v_k(X) \rangle = \langle v_k(Y) \rangle$ and Y would not be k -normal, or $\langle \tilde{Y} \rangle$ is a hyperplane in \mathbb{P}^{M+1} ; but this cannot occur, by the assumption on the degree, since $\tilde{Y} \subset \langle \tilde{Y} \rangle \cap \tilde{X}$. \square

In the next two theorems we collect the connections between lifting problem and k -normality and the known results about the lifting problem for varieties of dimension ≤ 3 . We will use the following notation: if X is a projective variety, we put

$$s(X) := \min\{k \mid h^0(\mathcal{I}_X(k)) \neq 0\}$$

and

$$\sigma(X) := \min\{k \mid h^0(\mathcal{I}_{X \cap H}(k)) \neq 0\}$$

where H is a general hyperplane. We will say that an integer number h is a *non-lifting level* for X if the restriction map $H^0(\mathcal{I}_X(h)) \rightarrow H^0(\mathcal{I}_{X \cap H}(h))$ is not surjective, when H is a general hyperplane.

Theorem 5. *Let $X \subset \mathbb{P}^n$ be an integral variety and S be a general hyperplane section of X . Let $k \geq 1$ be an integer number.*

- (1) *If X is k -normal, then all hypersurfaces of degree $k+1$ in \mathbb{P}^{n-1} containing S lift to a hypersurface of degree $k+1$ in \mathbb{P}^n containing X ;*
- (2) *if $k+1$ is a non-lifting level for X , then X is not k -normal.*

Theorem 6. *Let $X \subset \mathbb{P}^{n+2}$ ($n \geq 1$) be a nondegenerate lCM projective variety of codimension 2 and degree d .*

If $\sigma := \sigma(X) < s(X)$, then

- (1) *if $\dim(X) = 1$, $d \leq \sigma^2 + 1$;*

- (2) if $\dim(X) = 2$, $d \leq \sigma^2 - \sigma + 2$;
- (3) if $\dim(X) = 3$, $d \leq \sigma^2 - 2\sigma + 4$.

The proof of Theorem 5 follows immediately from the cohomology of the exact sequence

$$0 \rightarrow \mathcal{I}_X(k) \xrightarrow{\cdot H} \mathcal{I}_X(k+1) \rightarrow \mathcal{I}_S(k+1) \rightarrow 0.$$

For the proof of Theorem 6, we refer to the papers cited in the introduction.

Remark 1. Note that the second assertion of Theorem 5 cannot be reversed, because there exist varieties X such that $k+1$ is a lifting level (for a certain k), but nevertheless X is not k -normal. For instance, let X be a smooth rational curve in \mathbb{P}^3 of degree $d \geq 5$ and assume that it is contained in a quadric. Then X is not linearly normal but clearly 2 is a lifting level for X .

We now start to prove Theorem 2 of the introduction. Next proposition takes care of the threefolds with sectional genus zero.

Proposition 7. *Let $X' \subset \mathbb{P}^n$ be a rational normal scroll of dimension 3 (therefore of degree $n-2$), with $n \leq 8$; then a LCM linear projection X of X' to \mathbb{P}^5 is 2-normal. Moreover, $h^0(\mathcal{I}_X(3)) > 0$.*

Proof. By Lemma 4, applied to a general hyperplane section, it is sufficient to show the 2-normality for a projection of a 2-dimensional rational normal scroll Y' to a \mathbb{P}^4 . Let us call this projected surface $Y \subset \mathbb{P}^4$. We have $Y' = S(b_0, b_1)$, $0 \leq b_0 \leq b_1$, $b_0 + b_1 = n - 2$. Then the section C_0 (see [Har77], page 373) and the fiber f generate the Picard group of the ruled surface Y' . In particular, the hyperplane and canonical divisors are (numerically) $C_H = C_0 + b_1 f$ and $K = -2C_0 + (-2 - e)f$, where $-C_0^2 = e = b_1 - b_0$. Moreover $K^2 = 8$ (see again [Har77], pages 373–374).

In order to apply again Lemma 4, we will show that there exists a curve $C' \subset Y' \subset \mathbb{P}^{n-1}$ such that $\deg C' > \deg Y'$ and $C = \pi(C')$ is 2-normal and not contained in a quadric. Let us take $C' \in |C_0 + (b_1 + 9 - n)f|$; it is immediate to see, for example from the adjunction formula, that C' has arithmetic genus zero. Moreover, $|C'|$ is very ample (see [Har77], Theorem V.2.17(c)). So, if C' is general in its linear system, it is a smooth rational curve of degree $C' \cdot (C_0 + b_1 f) = b_0 + b_1 + 9 - n = 7$.

By Riemann-Roch $h^0(\mathcal{O}_C(2)) = 15 = h^0(\mathcal{O}_{\mathbb{P}^4}(2))$, so C is not 2-normal iff it is contained in a quadric. Let us consider then the hyperplane section C_H of Y ; if C_H is contained in a quadric, we can apply Theorem 6 to get that Y and X are contained in a quadric too. Therefore, X is arithmetically Cohen-Macaulay ([Kwa99], Theorem 2.1). So, let us suppose $h^0(\mathcal{I}_{C_H}(2)) = 0$.

If $n = 8$, C is linearly equivalent to $C_H \cup f$. Then,

$$\frac{\mathcal{I}(C_H)}{\mathcal{I}(C_H) \cap \mathcal{I}(f)} \cong \frac{\mathcal{I}(C_H) + \mathcal{I}(f)}{\mathcal{I}(f)} \cong \frac{\mathcal{I}(C_H \cap f)}{\mathcal{I}(f)}$$

and so the following sequence

$$0 \rightarrow \mathcal{I}_{C_H \cup f} \rightarrow \mathcal{I}_{C_H} \rightarrow \mathcal{I}_{P|f} \rightarrow 0,$$

where $P = C_H \cap f$, is exact. We have that $\mathcal{I}_{P|f} \cong \mathcal{O}_f(-1)$, and since C_H is degenerate and it is not contained in a quadric, we obtain

$$(1) \quad 0 \rightarrow H^0(\mathcal{I}_{C_H \cup f}(2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \rightarrow H^0(\mathcal{O}_f(1)) \rightarrow H^1(\mathcal{I}_{C_H \cup f}(2)) \rightarrow 0.$$

Now, $h^0(\mathcal{O}_{\mathbb{P}^4}(1)) = 5$, $h^0(\mathcal{O}_f(1)) = 2$ and $h^1(\mathcal{I}_{C_H \cup f}(2)) \geq h^1(\mathcal{I}_C(2))$ by semicontinuity (Theorem III.12.8 of [Har77]). If, by contradiction, $h^1(\mathcal{I}_C(2)) \neq 0$, then $h^1(\mathcal{I}_{C_H \cup f}(2)) \neq 0$, so $h^0(\mathcal{I}_{C_H \cup f}(2)) > 3$, hence there exists at least one irreducible quadric in \mathbb{P}^4 containing $C_H \cup f$, in particular C_H , a contradiction. Therefore, C is 2-normal and not contained in a quadric, and by Lemma 4 X is 2-normal.

If $n = 7$, C is linearly equivalent to $C_H \cup f_1 \cup f_2$, where f_1 and f_2 are two fibers. Arguing as in the previous case, we obtain an exact sequence

$$0 \rightarrow H^0(\mathcal{I}_{C_H \cup f_1 \cup f_2}(2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \rightarrow \begin{array}{c} H^0(\mathcal{O}_{f_1}(1)) \\ \oplus \\ H^0(\mathcal{O}_{f_2}(1)) \end{array} \rightarrow H^1(\mathcal{I}_{C_H \cup f_1 \cup f_2}(2)) \rightarrow 0.$$

If $h^1(\mathcal{I}_C(2)) \neq 0$, this time we get $h^0(\mathcal{I}_{C_H \cup f_1 \cup f_2}(2)) > 1$ and, since $\langle f_1, f_2 \rangle \cong \mathbb{P}^3$, we get again an irreducible quadric which contains C_H .

If $n = 6$, C is linearly equivalent to $C_H \cup f_1 \cup f_2 \cup f_3$; in this case, it suffices to observe that, by semicontinuity, $h^0(\mathcal{I}_{C_H \cup f_1 \cup f_2 \cup f_3}(2)) \geq h^0(\mathcal{I}_C(2))$, so, if C is contained in a quadric, also $C_H \cup f_1 \cup f_2 \cup f_3$ is contained in a quadric. This quadric is irreducible since the three fibers generate \mathbb{P}^4 . So we get again a contradiction.

The last assertion follows from the fact that $h^0(\mathcal{I}_{C_H}(3)) > 0$ by Riemann Roch, and by the fact that X and Y are quadratically normal. \square

1.1. Threefolds of degree 5 in \mathbb{P}^5 . We shall now prove that a LCM threefold X of degree $d = 5$ is always contained either in a quadric or in a cubic and is 2-normal.

Theorem 8. *Let X be a LCM threefold of degree 5 and sectional genus π in \mathbb{P}^5 not contained in a quadric; then X is contained in a cubic and is 2-normal. Moreover, if $\pi = 1$ then $h^0(\mathcal{I}_X(3)) = 5$.*

Proof. If the geometric genus of the curve section C is zero, then we apply Proposition 7.

If $\pi = 2$, then C is a curve of maximal genus, and therefore it is arithmetically Cohen-Macaulay (aCM for short), and so also the surface section S and X are aCM.

If $\pi = 1$, let us consider the long exact sequence of cohomology

$$(2) \quad 0 \rightarrow H^0(\mathcal{I}_X(3)) \rightarrow H^0(\mathcal{I}_S(3)) \rightarrow H^1(\mathcal{I}_X(2)) \rightarrow \cdots$$

The intersection of X with a general plane is formed by 5 points, which are contained in a conic. If this conic lifts to a quadric containing the curve section of X , by Theorem 6 for surfaces and threefolds it lifts also to a quadric containing X . So it does not lift, then the curve section $C = H \cap H' \cap X$ has genus one, and it is bilinked to a couple of skew lines ℓ_1 and ℓ_2 : $C \underset{(3,3)}{\sim} C_4 \underset{(2,3)}{\sim} \ell_1 \cup \ell_2$, where C_4 is a rational quartic.

Now, from

$$0 \rightarrow \mathcal{I}_S(1) \rightarrow \mathcal{I}_S(2) \rightarrow \mathcal{I}_C(2) \rightarrow 0,$$

since $H^0(\mathcal{I}_C(2)) = 0$, we get

$$(3) \quad 0 \rightarrow H^1(\mathcal{I}_S(1)) \rightarrow H^1(\mathcal{I}_S(2)) \rightarrow H^1(\mathcal{I}_C(2)) \rightarrow \cdots$$

By Severi's Theorem $H^1(\mathcal{I}_S(1)) = 0$. But also $H^1(\mathcal{I}_C(2)) = 0$: indeed by

$$0 \rightarrow \mathcal{I}_C(2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(2) \rightarrow \mathcal{O}_C(2) \rightarrow 0,$$

we get

$$(4) \quad 0 \rightarrow H^0(\mathcal{I}_C(2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \rightarrow H^0(\mathcal{O}_C(2)) \rightarrow H^1(\mathcal{I}_C(2)) \rightarrow 0$$

and, using Riemann-Roch, we conclude that $h^0(\mathcal{I}_C(2)) = h^1(\mathcal{I}_C(2)) = 0$.

Then $H^1(\mathcal{I}_S(2)) = 0$ so $h^0(\mathcal{I}_S(3)) = h^0(\mathcal{I}_C(3))$. Now, $h^0(\mathcal{I}_C(3)) = 5$ (by mapping cone), so also $h^0(\mathcal{I}_S(3)) = 5$.

From the exact sequence

$$(5) \quad 0 \rightarrow H^0(\mathcal{I}_X(1)) \rightarrow H^0(\mathcal{I}_X(2)) \rightarrow H^0(\mathcal{I}_S(2)) \rightarrow H^1(\mathcal{I}_X(1)) \rightarrow H^1(\mathcal{I}_X(2)) \rightarrow 0$$

and $H^0(\mathcal{I}_X(1)) = 0$, we obtain $H^1(\mathcal{I}_X(2)) \cong H^1(\mathcal{I}_X(1)) = 0$; and analogously, by

$$(6) \quad 0 \rightarrow H^0(\mathcal{I}_X(2)) \rightarrow H^0(\mathcal{I}_X(3)) \rightarrow H^0(\mathcal{I}_S(3)) \rightarrow H^1(\mathcal{I}_X(2)) \rightarrow 0$$

we conclude that $h^0(\mathcal{I}_X(3)) = 5$. \square

1.2. Threefolds of degree 6 in \mathbb{P}^5 . By Proposition 7 it suffices to analyze the cases of positive sectional genus π . By Castelnuovo bound, $\pi \leq 4$. If $\pi = 4$, then the curve section is aCM, so also the threefold X is aCM.

If $\pi = 3$, then with computations similar to those made in the proof of Theorem 8 we conclude that also in this case X is quadratically normal and, if it is not contained in a quadric, it is contained in a cubic.

We will show in the next section that there do exist threefolds of degree 6 and sectional genus one which are not quadratically normal (necessarily singular); so we analyze now the case of sectional genus two:

Proposition 9. *Let X be a threefold of degree 6 and sectional genus two. Then $h^0(\mathcal{I}_X(3)) > 0$ and $h^1(\mathcal{I}_X(2)) \leq 1$.*

Proof. Let S, C and Γ be, respectively, the general linear sections of dimensions 2, 1 and 0.

If $h^0(\mathcal{I}_\Gamma(2)) > 0$, by Theorem 6, we have as usual that also X is contained in a quadric and is quadratically normal, and we are done.

Let us suppose now $h^0(\mathcal{I}_\Gamma(2)) = 0$. Then \mathcal{I}_Γ has the following minimal free resolution:

$$0 \rightarrow 3\mathcal{O}_{\mathbb{P}^2}(-4) \rightarrow 4\mathcal{O}_{\mathbb{P}^2}(-3) \rightarrow \mathcal{I}_\Gamma \rightarrow 0,$$

with $h^0(\mathcal{I}_\Gamma(3)) = 4$. From $h^0(\mathcal{I}_C(2)) = 0$ and Riemann-Roch, we have $h^1(\mathcal{I}_C(1)) = h^1(\mathcal{I}_C(2)) = 1$. Moreover, from [GLP83], $h^1(\mathcal{I}_C(k)) = 0 \ \forall k \geq 3$. In particular, $h^0(\mathcal{I}_C(3)) = 3$.

Note that $h^1(\mathcal{I}_S(2)) \leq 1$, because we have

$$0 \rightarrow H^1(\mathcal{I}_S(1)) \rightarrow H^1(\mathcal{I}_S(2)) \rightarrow H^1(\mathcal{I}_C(2)) \rightarrow \dots$$

with S linearly normal and $h^1(\mathcal{I}_C(2)) = 1$.

From the exact sequence

$$(7) \quad 0 \rightarrow H^0(\mathcal{I}_S(3)) \rightarrow H^0(\mathcal{I}_C(3)) \rightarrow H^1(\mathcal{I}_S(2)) \rightarrow H^1(\mathcal{I}_S(3)) \rightarrow 0$$

we can deduce now that $h^0(\mathcal{I}_S(3)) \geq 2$. Finally, let us consider the sequence

$$0 \rightarrow H^0(\mathcal{I}_X(3)) \rightarrow H^0(\mathcal{I}_S(3)) \rightarrow H^1(\mathcal{I}_X(2)) \rightarrow \dots$$

Since $h^1(\mathcal{I}_X(2)) \leq h^1(\mathcal{I}_S(2)) \leq 1$, we get $h^0(\mathcal{I}_X(3)) > 0$. \square

We have concluded in this way the proof of Theorem 2 of the introduction.

Corollary 10. *The congruence of the 4-secant lines of a lCM threefold of degree 5 or of degree 6 with $\pi \neq 1$ cannot have positive order.*

Remark 2. We observe that we can slightly weaken the hypothesis of Theorem 2 in the case $d = 6$. In fact in Proposition 7 we have proved that if the geometric sectional genus is zero, then X is quadratically normal and contained in a cubic.

Remark 3. In [Koe92], it has been shown that if a lCM threefold $X \subset \mathbb{P}^5$ is contained in a cubic hypersurface which is also lCM, then X is aCM. Moreover, if X is smooth, the assumption lCM on the cubic hypersurface can be removed.

To state the following proposition, we need to recall two definitions about first order congruences of lines B in \mathbb{P}^n from [DP05], and precisely those of *parasitical scheme* and of *pure focal locus*. A parasitical scheme is a subscheme of the focal locus, of dimension at least two, covered by lines of B and that is not met by a

general line of the congruence. The *pure focal locus* is the union of the components of the focal locus which are not parasitical. Note that therefore the general lines of B are lines meeting $n - 1$ times the pure fundamental locus (see [DP05], Proposition 2.1).

Proposition 11. *Let B be a first order congruence of lines in \mathbb{P}^5 such that its pure focal locus X is an irreducible threefold of degree six. Then, the sectional genus of X is one and the multidegree of B is $(1, 3, a)$, with $a \leq 3$.*

Proof. X must be of sectional genus one, by Corollary 10. If $C \subset \mathbb{P}^3$ is a smooth connected sextic curve of genus one, then the expected number of 4-secant lines is 3, see [LeB82].

Moreover we have $h^0(\mathcal{I}_C(3)) = 2$, since C is cubically normal by [GLP83]. From sequence (7) we deduce $h^0(\mathcal{I}_S(3)) = 1$, since otherwise S , and therefore X , would be quadratically normal, therefore, the second multidegree is 3.

The third multidegree a is necessarily ≤ 3 , since we have observed that the expected value is 3, but some of these 3 lines could be parasitical. \square

We will see in Section 2 an example with a parasitical line. We will also show the existence of congruences of multidegree $(1, 3, a)$ for any $0 \leq a \leq 3$.

2. LINEAR AND QUADRATIC CONGRUENCES

We recall that a *congruence of lines* in \mathbb{P}^5 is a flat family of lines of dimension 4. A *linear congruence* is a congruence obtained by a linear section—with a linear space Δ of dimension 10—of the Grassmannian $\mathbb{G}(1, 5) \subset \mathbb{P}^{14}$. Linear congruences in \mathbb{P}^5 are studied and classified in [DM05].

We recall that such a classification is obtained considering $\check{\mathbb{G}}(1, 5) \subset \check{\mathbb{P}}^{14}$ which is a cubic hypersurface, the *Pfaffian*, intersected with $\check{\Delta} \cong \mathbb{P}^3$. Then, we have classified the linear congruences considering the surface $Y := \check{\mathbb{G}}(1, 5) \cap \check{\Delta}$ and its possible positions in $\check{\mathbb{G}}(1, 5)$ and with $\text{Sing}(\check{\mathbb{G}}(1, 5)) \cong \mathbb{G}(3, 5)$.

Among all the cases considered, particularly interesting is the case 5.2.1 of [DM05]: in this situation Y is reducible, with $Y = \pi \cup Q$, where π is a plane, Q is a quadric, and $Y \cap \mathbb{G}(3, 5) = \emptyset$. Let

$$\gamma: \check{\mathbb{G}}(1, 5) \dashrightarrow \mathbb{G}(1, 5)$$

be the Gauss map. Its restriction to the plane π is a degree 2 Veronese morphism, hence its image is a Veronese surface. It parametrizes the secant lines of a skew cubic curve C embedded in a three dimensional linear space $L \subset \mathbb{P}^5$.

Let us recall that the linear spaces contained in $\check{\mathbb{G}}(1, 5) \setminus \mathbb{G}(3, 5)$ can be interpreted as linear spaces of 6×6 skew-symmetric matrices of constant rank 4. They are classified in [MM05]. In particular, the planes are distributed in 4 \mathbb{PGL}_6 -orbits, all of dimension 26, and the one we are considering is generated for instance by the plane

$$(8) \quad \pi_t = \begin{pmatrix} 0 & a & b & c & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & a \\ & & & 0 & 0 & b \\ & & & & 0 & c \\ & & & & & 0 \end{pmatrix}$$

$$(9) \quad = ah_1 + bh_2 + ch_3.$$

Dually, $\tilde{\pi}_t = H_1 \cap H_2 \cap H_3$ is a 11-dimensional linear space, which is tangent to the Grassmannian along the Veronese surface $\gamma(\pi_t)$.

In terms of coordinates, in the Plücker embedding, by (8), the ideal of $\tilde{\pi}_t$ is

$$(10) \quad I(\tilde{\pi}_t) = \langle p_{01} + p_{25}, p_{02} + p_{35}, p_{03} + p_{45} \rangle.$$

Let us denote by Γ the intersection

$$(11) \quad \Gamma := \tilde{\pi}_t \cap \mathbb{G}(1, 5).$$

It is a 5-dimensional family of lines.

We note that in this case the Veronese surface parametrizes the secant lines of the twisted cubic C contained in the 3-dimensional subspace L of equations $x_0 = x_5 = 0$ defined by

$$(12) \quad \text{rk} \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} < 2.$$

The lines of Γ can be described as follows:

Lemma 12. *Let us denote by Γ_P the lines of Γ through P ; then*

- (1) *if $P \in \mathbb{P}^5 \setminus L$, Γ_P forms a pencil of lines in a plane that intersects C in only one point;*
- (2) *if $P \in L \setminus C$, Γ_P is the star of lines through P contained in L ;*
- (3) *if $P \in C$, $\dim(\Gamma_P) = 3$ and Γ_P generates a hyperplane of \mathbb{P}^5 (which contains L).*

Proof. If $P = (a_0 : \dots : a_5) \in \mathbb{P}^5$, then, by (10), and recalling that $p_{ij} = x_i a_j - x_j a_i$, the lines through P belonging to Γ are contained in the linear space defined by

$$(13) \quad a_1 x_0 - a_0 x_1 + a_5 x_2 - a_2 x_5 = 0$$

$$(14) \quad a_2 x_0 - a_0 x_2 + a_5 x_3 - a_3 x_5 = 0$$

$$(15) \quad a_3 x_0 - a_0 x_3 + a_5 x_4 - a_4 x_5 = 0.$$

Now, the matrix of the coefficients

$$(16) \quad A = \begin{pmatrix} a_1 & -a_0 & a_5 & 0 & 0 & -a_2 \\ a_2 & 0 & -a_0 & a_5 & 0 & -a_3 \\ a_3 & 0 & 0 & -a_0 & a_5 & -a_4 \end{pmatrix}$$

has maximal rank if and only if $P \in \mathbb{P}^5 \setminus L$. In fact, it is an easy exercise to show that

$$(17) \quad \text{rk}(A) < 3 \iff P \in L$$

and, if $a_0 = a_5 = 0$

$$(18) \quad \text{rk}(A) < 2 \iff P \in C,$$

and the thesis follows. □

Now, in order to obtain a congruence B , we have to intersect Γ with a hyper-surface. Since we are interested in first order congruences, we may need that the congruence B splits in some components.

We have performed the computations in the following examples with the help of Macaulay2 [M2].

We note first that $\mathbb{G}(1, L)$ is the intersection of $\mathbb{G}(1, 5)$ with the following 5-space:

$$(19) \quad \Pi_L = V(p_{01}, p_{02}, p_{03}, p_{04}, p_{05}, p_{15}, p_{25}, p_{35}, p_{45})$$

and the hyperplane $H_L := V(p_{05})$ is the only one which is tangent to $\mathbb{G}(1, 5)$ along $\mathbb{G}(1, L)$.

Example 1. We first intersect Γ with a general hyperplane H , in particular we request that $H \not\supset \mathbb{G}(1, L)$. This is case 5.2.1 of [DM05]. In this case the congruence B is irreducible, and the focal locus is $F = X \cup L'$, where L' is a scheme whose support is L , having C as an embedded component, and X is a sextic threefold, with Hilbert polynomial $P_X(t) = t^3 + 3t^2 + 2$. $X \cap L$ is a quartic surface with C as singular locus, and L' is parasitical: therefore the lines of B are the 4-secants to X . The multidegree of B is $(1, 3, 2)$, so the hyperplane section of X , S , is contained in a cubic, while X is not. The singular locus of X is just the twisted cubic C . The sectional genus is one: hence applying the quadruple point formula for a curve in \mathbb{P}^3 (see [LeB82]), we see that the curve section C_H of X has three quadrisecant lines: two are the lines of the congruence, and the third is $H \cap L$, *i.e.* a parasitical line.

X is ICM: in fact, its ideal sheaf \mathcal{I}_X has a minimal free resolution of the form

$$(20) \quad 0 \rightarrow \mathcal{O}(-8) \rightarrow \mathcal{O}^6(-7) \xrightarrow{A} \mathcal{O}^{16}(-6) \rightarrow \mathcal{O}^{22}(-5) \rightarrow \mathcal{O}^{12}(-4) \rightarrow \mathcal{I}_X$$

and the points where X is not Cohen-Macaulay are the ones defined by the first nonzero Fitting ideal of A (see for example [RS91]). It results that the map A has a nonzero kernel, and the 5×5 -minors of A define an irrelevant ideal, therefore X is ICM.

As an explicit example, let us take $H = V(p_{05} + p_{12} + p_{14} + p_{34})$ so the quadric Q , component of the cubic surface Y , has equation $a^2 + b^2 + c^2 + d^2 - ac$, *i.e.* the Pfaffian divided by d of the matrix

$$(21) \quad A(d) = \begin{pmatrix} 0 & a & b & c & 0 & d \\ & 0 & d & 0 & d & 0 \\ & & 0 & 0 & 0 & a \\ & & & 0 & d & b \\ & & & & 0 & c \\ & & & & & 0 \end{pmatrix}.$$

Finally, we observe that the lines of B passing through a general point $P \in X$ form a planar pencil since B is a linear congruence (see Proposition 4.2 of [DM05]).

Example 2. If we intersect Γ with a hyperplane H such that $H \supset \mathbb{G}(1, L)$ but H is not tangent to $\mathbb{G}(1, 5)$ along $\mathbb{G}(1, L)$ (*i.e.* $H \neq H_L$), then $B = B' \cup \mathbb{G}(1, L)$; the multidegree of B' is then $(1, 3, 1)$. The skew-symmetric matrix associated to $\tilde{\Delta}$ (where $\Delta = \Gamma \cap H$ as usual) is of the form

$$(22) \quad A(a_0, \dots, a_8) = \begin{pmatrix} 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ & 0 & 0 & 0 & 0 & a_5 \\ & & 0 & 0 & 0 & a_6 \\ & & & 0 & 0 & a_7 \\ & & & & 0 & a_8 \\ & & & & & 0 \end{pmatrix}.$$

The \mathbb{P}^8 of these matrices is the embedded tangent space to $\mathbb{G}(3, 5)$ at the point corresponding to the hyperplane $H_L = V(p_{05})$ of \mathbb{P}^{14} , *i.e.* $\mathbb{T}_L \mathbb{G}(3, 5)$. In particular, since $\text{rk}(A(a_0, \dots, a_8)) \leq 4$, the 3-space $\tilde{\Delta}$ is contained in $\check{\mathbb{G}}(1, 5)$, *i.e.* $\tilde{\Delta}$ is a \mathbb{P}^3 of matrices of rank at most 4. Since the intersection of $\mathbb{G}(3, 5)$ with its tangent space at a point P is a cone of vertex P over $\mathbb{P}^1 \times \mathbb{P}^3$, it has degree 4 and dimension 5, hence, for general Δ , $\tilde{\Delta} \cap \mathbb{G}(3, 5)$ is given by 4 points. These four points correspond to 4 \mathbb{P}^3 's in \mathbb{P}^5 , that result to be the components of the pure focal locus of B' , and the lines of B' are the quadrisecants of these \mathbb{P}^3 's. We recall that a point in $\mathbb{G}(3, 5) \cap \mathbb{T}_L \mathbb{G}(3, 5)$ represents a \mathbb{P}^3 intersecting L along a 2-plane. This implies that the 4 \mathbb{P}^3 's are not in general position, but they intersect two by two in a line of L and three by three at a point in L . L turns out to be a parasitical component of the

focal locus of B' . This parasitical component has multiplicity two since through every point of L there passes a 2-dimensional family of lines of B' .

Remark 4. This example was not considered in [DM05], where only the case of a general hyperplane H was studied. Indeed in that article, in Proposition 5.3, we erroneously claimed that, if a 3-space is contained in $\mathbb{G}(1, 5)$, then the corresponding focal locus has dimension > 3 . The same error is contained in [FM02], proof of Theorem 4.3 (as a matter of facts, Proposition 5.3 of [DM05] is a particular case of Theorem 4.3). The present case is the only exception to the quoted Theorem of [FM02] (see [FM08]).

An explicit example can be taken with $H = V(p_{04} + p_{15})$, which gives the matrix

$$(23) \quad A(d) = \begin{pmatrix} 0 & a+d & b & c & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & a \\ & & & 0 & 0 & b \\ & & & & 0 & c+d \\ & & & & & 0 \end{pmatrix}.$$

The four singular hyperplanes are given by $H_0 = V(p_{25} - p_{45})$, $H_1 = V(p_{02} + p_{03} + p_{25} + p_{35})$, $H_2 = V(p_{02} - p_{03} - p_{25} + p_{35})$, and $H_3 = V(p_{01} - p_{03})$. They correspond to the four \mathbb{P}^3 's, respectively, $L_0 = V(x_5, x_2 - x_4)$, $L_1 = V(x_0 - x_5, x_2 + x_3)$, $L_2 = V(x_0 + x_5, x_2 - x_3)$, and $L_3 = V(x_0, x_1 - x_3)$. $\mathcal{I}(B')$ modulo $\mathcal{I}(\mathbb{G}(1, 5))$ is

$$(24) \quad A = (p_{01} + p_{45}, p_{02} + p_{35}, p_{03} + p_{45}, p_{25} - p_{45}, \\ p_{13}p_{23} - p_{12}p_{24} - p_{23}p_{24} + p_{13}p_{34}).$$

Example 3. If we intersect Γ with the hyperplane H_L , then—set-theoretically— $B = B'' \cup \mathbb{G}(1, L)$, but $\mathbb{G}(1, L)$ has to be considered with multiplicity two; the multidegree of B'' is then $(1, 3, 0)$. The focal locus of B'' is in this case a scheme having support on L , with C as an embedded component; B'' is geometrically given in the following way: consider an isomorphism from the pencil of hyperplanes containing L , \mathbb{P}_L^1 , to C , $\phi: \mathbb{P}_L^1 \rightarrow C$. Then

$$(25) \quad B'' = \overline{\cup_{H \in \mathbb{P}_L^1} \mathbb{P}_{\phi(H)}^2}$$

where $\mathbb{P}_{\phi(H)}^2$ is the star of lines in H with support in the point $\phi(H)$.

In terms of coordinates, $\mathcal{I}(B'')$ modulo $\mathcal{I}(\mathbb{G}(1, 5))$ is

$$(26) \quad A_0 = (p_{01} + p_{25}, p_{02} + p_{35}, p_{03} + p_{45}, p_{05}, \\ p_{15}p_{34} - p_{25}p_{24} + p_{35}p_{23}, \\ p_{23}^3 - p_{13}p_{23}p_{24} + p_{12}p_{24}^2 + p_{13}^2p_{34} - 2p_{12}p_{23}p_{34} - p_{12}p_{14}p_{34}),$$

i.e. B'' is a nonproper intersection of a quadratic and a cubic complex with Γ .

Next theorem collects the results obtained in the three examples above.

Theorem 13. *Let Γ be the intersection of $\mathbb{G}(1, 5)$ with the dual of the plane π_t described by equation (8). Let B denote the congruence of lines $\Gamma \cap H$, where H is a hyperplane of \mathbb{P}^{14} . Let L be the 3-space of equations $x_0 = x_5 = 0$ and $C \subset L$ be the skew cubic defined by Equation 12.*

- (1) *If H does not contain $\mathbb{G}(1, L)$, then B is irreducible of multidegree $(1, 3, 2)$. Its focal locus is $F = X \cup L'$, where L' is a non-reduced scheme with $L'_{\text{red}} = L$, and X is a lCM threefold of degree 6 with $\pi = 1$, and with Hilbert polynomial $P_X(t) = t^3 + 3t^2 + 2$. $X \cap L$ is a quartic surface which is singular along the cubic curve C . L' is a parasitical component of F .*

- (2) If H is a general hyperplane containing $\mathbb{G}(1, L)$, then $B = \mathbb{G}(1, L) \cup B'$, where B' is a congruence of multidegree $(1, 3, 1)$. The focal locus of B' is the union of four 3-spaces and a double structure on L , which is a parasitical component. The four 3-spaces intersect two by two along a line of L and three by three at a point.
- (3) If $H = H_L$, the hyperplane which is tangent to $\mathbb{G}(1, 5)$ along $\mathbb{G}(1, L)$, then $B = 2\mathbb{G}(1, L) \cup B''$, where B'' is a congruence of multidegree $(1, 3, 0)$. The focal locus of B'' is a non-reduced scheme of support L , having C as embedded component.

We will consider now the quadratic complexes containing Γ , always with the aim of finding first order congruences. So we take $\Gamma \cap Q$, where Q is a quadratic complex. If Q is general, then $\Gamma \cap Q$ is an irreducible congruence of order 2, hence the focal locus has dimension 4 by Theorem 2.1 of [DP04]. In order to get a reducible congruence, having an irreducible component of order one, we require that Q contains both B'' and $\mathbb{G}(1, L)$. We will prove now that quadrics of this type do exist.

Theorem 14. *There exists a family of dimension 12 of quadrics of \mathbb{P}^{14} containing $\mathbb{G}(1, L) \cup B''$. If Q is such a quadric, then $\Gamma \cap Q = \mathbb{G}(1, L) \cup B'' \cup B$, where B is a congruence of multidegree $(1, 3, 3)$, that is irreducible for general Q . The focal locus of B is a threefold of degree 6 with $\pi = 1$.*

Proof. The equation q of the quadric Q must be chosen in A_0 of Equation (26), i.e. it must be a combination of the polynomials $p_{01} + p_{25}, p_{02} + p_{35}, p_{03} + p_{45}, p_{05}$, and $p_{15}p_{34} - p_{25}p_{24} + p_{35}p_{23}$. Let us observe that in this case Q automatically contains the linear span of $\mathbb{G}(1, L)$. Let us consider $Q \cap \tilde{\pi}_t$: its ideal, modulo $\mathcal{I}(\tilde{\pi}_t)$, is generated by p_{05} and $p_{12}p_{04} - p_{23}p_{35} + p_{13}p_{45}$, therefore the family of congruences we are looking for is the residual of $B'' \cup \mathbb{G}(1, L)$ in the intersection

$$(27) \quad Q(d; a_1, \dots, a_6; b_1, \dots, b_5; c) \cap \Gamma = V(p_{01} + p_{25}, p_{02} + p_{35}, p_{03} + p_{45}, q)$$

where

$$\begin{aligned} q = q(d; a_1, \dots, a_6; b_1, \dots, b_5; c) := & -d(p_{15}p_{34} - p_{25}p_{24} + p_{35}p_{23}) + \\ & + p_{05}(a_1p_{12} + a_2p_{13} + a_3p_{14} + a_4p_{23} + a_5p_{24} + a_6p_{34} + \\ & - (b_1p_{15} + b_2p_{25} + b_3p_{35} + b_4p_{45}) + b_5p_{04} - cp_{05}). \end{aligned}$$

The multidegree of B is obtained simply by subtracting to the one of $\Gamma \cap Q$, which is $(2, 6, 4)$, those of $\mathbb{G}(1, L)$ and of B'' . The fact that in general B is an irreducible congruence has been checked by a direct computation with Macaulay 2. By Corollary 10 it follows that the focal locus has degree 6 and sectional genus 1. \square

Remark 5. We observe that Q can have rank equal to 8, 7, 6 or (at most) 2. It has rank 6 if and only if $a_i = b_j = c = 0, \forall i, j$ and $d \neq 0$. If $d = 0$, the quadric has rank at most 2. The general case of rank 8 is obtained if and only if $d \neq 0$ and at least one among a_1, a_2, a_3, b_4, b_5 is not zero. Finally, if $d \neq 0$ and $a_1 = a_2 = a_3 = b_4 = b_5 = 0$ (but at least one of the other terms is different from zero), the quadric has rank 7.

Clearly, the case of rank 2 corresponds to reducible quadratic congruences (which contain the tangent linear congruence of Theorem 13, case (3)).

We will describe in detail the focal locus of the congruence B in the next section.

3. ON THE COMPLETELY EXCEPTIONAL MONGE-AMPÈRE EQUATIONS

For the definitions and generalities about systems of conservation laws and their correspondence with congruences of lines, we refer to [AF01], see also [DM05]. An important class of strictly hyperbolic PDE's of conservation laws are the *completely*

exceptional Monge-Ampère equations, which are introduced and studied in [Boi92]. It is shown there that they are *linearly degenerate*, which means that the eigenvalues of the Jacobian matrix associated to the system are constant along the rarefaction curves. In [AF01], it is asserted that this class is even a *T-system*, *i.e.* that the rarefaction curves are lines, and it is suggested that this could be an example of a non-linear *T-system*. We will show that this is the case for systems of the fourth order; a similar statement can be proven analogously in general.

The completely exceptional Monge-Ampère systems of the fourth order are defined as follows (see [AF01], Section 5). Introduce the *Henkel (or persymmetric) matrix*

$$H := \begin{pmatrix} \frac{\partial^4 u}{\partial x^4} & \frac{\partial^4 u}{\partial x^3 \partial t} & \frac{\partial^4 u}{\partial x^2 \partial t^2} \\ \frac{\partial^4 u}{\partial x^3 \partial t} & \frac{\partial^4 u}{\partial x^2 \partial t^2} & \frac{\partial^4 u}{\partial x \partial t^3} \\ \frac{\partial^4 u}{\partial x^2 \partial t^2} & \frac{\partial^4 u}{\partial x \partial t^3} & \frac{\partial^4 u}{\partial t^4} \end{pmatrix}$$

and consider the PDE of the fourth order

$$(28) \quad d \det(H) + a_1 \left(\left(\frac{\partial^4 u}{\partial x^4} \frac{\partial^4 u}{\partial x^2 \partial t^2} - \left(\frac{\partial^4 u}{\partial x^3 \partial t} \right)^2 \right) \right) + \cdots + b_1 \frac{\partial^4 u}{\partial x^4} + \cdots + c = 0.$$

i.e. a linear combination of the minors of all orders of H (where we suppose $d \neq 0$).

After the introduction of the new variables $(u_1, u_2, u_3, u_4, u_5)$ such that

$$u_1 = \frac{\partial^4 u}{\partial x^4}, \quad u_2 = \frac{\partial^4 u}{\partial x^3 \partial t}, \quad u_3 = \frac{\partial^4 u}{\partial x^2 \partial t^2}, \quad u_4 = \frac{\partial^4 u}{\partial x \partial t^3}, \quad u_5 = \frac{\partial^4 u}{\partial t^4},$$

Equation (28) becomes

$$(29) \quad d \det \begin{pmatrix} u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \\ u_3 & u_4 & u_5 \end{pmatrix} + a_1(u_1 u_3 - u_2^2) + \cdots + b_1 u_1 + \cdots + c = 0,$$

moreover

$$(30) \quad (u_1)_t = (u_2)_x, \quad (u_2)_t = (u_3)_x, \quad (u_3)_t = (u_4)_x, \quad (u_4)_t = (u_5)_x.$$

This is a system of conservation laws, and the corresponding congruence B_{MA} in \mathbb{P}^5 , according to the Agafonov-Ferapontov construction [AF01] is (in non-homogeneous coordinates (y_0, \dots, y_4))

$$\begin{aligned} y_1 &= u_1 y_0 - u_2 \\ y_2 &= u_2 y_0 - u_3 \\ y_3 &= u_3 y_0 - u_4 \\ y_4 &= u_4 y_0 - u_5 \end{aligned}$$

together with Equation (29), or, using projective coordinates $(y_0 : \dots : y_5)$ in \mathbb{P}^5 and $(u_0 : \dots : u_5)$ as parameters for the lines in B_{MA} , it is given by

$$\begin{aligned} u_0 y_1 &= u_1 y_0 - u_2 y_5 \\ u_0 y_2 &= u_2 y_0 - u_3 y_5 \\ u_0 y_3 &= u_3 y_0 - u_4 y_5 \\ u_0 y_4 &= u_4 y_0 - u_5 y_5. \end{aligned}$$

Theorem 15. *The congruence B_{MA} corresponding to a completely exceptional Monge-Ampère system coincides with a congruence B of Theorem 14. In particular B_{MA} has multidegree $(1, 3, 3)$.*

Proof. To obtain the equations of B_{MA} in the Plücker embedding, we notice that the line corresponding to the parameters $(u_0 : \dots : u_5)$ joins the points $(u_0 : \dots : u_4 : 0)$ and $(0 : -u_2 : \dots : -u_5 : u_0)$. Hence we can compute its Plücker coordinates taking the 2×2 -minors of the matrix

$$M = \begin{pmatrix} u_0 & u_1 & \cdots & u_4 & 0 \\ 0 & -u_2 & \cdots & -u_5 & u_0 \end{pmatrix}.$$

We see immediately that they satisfy the equations in (27) of $\Gamma \cap Q = \mathbb{G}(1, L) \cup B'' \cup B$. In particular, Equation (29), made homogeneous of degree 4, becomes the equation of the quadric Q . The components $\mathbb{G}(1, L)$ and B'' are both contained in the hyperplane $V(p_{05})$, but the coordinate p_{05} of a line in B_{MA} has $p_{05} \neq 0$. This implies that $B_{MA} = B$, in particular the multidegree of B_{MA} is $(1, 3, 3)$. \square

Remark 6. Note that the degree of B in the Plücker embedding is 16.

By the above description of the congruences B and B_{MA} , we can find their focal loci in the following way. By definition, the focal locus is the image of the ramification divisor of the map $f: \Lambda \rightarrow \mathbb{P}^5$, where $\Lambda \subset B \times \mathbb{P}^5$ is the incidence variety and f is the restriction of the projection. We can work locally, so we can choose local coordinates u_1, \dots, u_4 on an open (analytical) subset of B (since by (29) it can be written as $u_5 = g(u_1, \dots, u_4)$) and y_0, \dots, y_4 as affine coordinates on \mathbb{P}^5 , so y_0, u_1, \dots, u_4 are local coordinates on an open subset of Λ . With this choice, the Jacobian matrix of f is

$$Df = \begin{pmatrix} \partial f / \partial y_0 \\ \partial f / \partial u_1 \\ \vdots \\ \partial f / \partial u_4 \end{pmatrix} = \begin{pmatrix} 1 & u_1 & u_2 & u_3 & u_4 \\ 0 & y_0 & 0 & 0 & -g_1 \\ 0 & -1 & y_0 & 0 & -g_2 \\ 0 & 0 & -1 & y_0 & -g_3 \\ 0 & 0 & 0 & -1 & y_0 - g_4 \end{pmatrix}$$

where $g_i = \partial g / \partial y_i$. Then, the ramification divisor is just the closure of $V(\det(Df)) \cap \Lambda$ and to find the focal locus X we have to eliminate the variables u_i 's. Actually, the computations were made by observing that

$$(31) \quad \det(Df) = \det \begin{pmatrix} y_0 & 0 & 0 & 0 & h_1 \\ -1 & y_0 & 0 & 0 & h_2 \\ 0 & -1 & y_0 & 0 & h_3 \\ 0 & 0 & -1 & y_0 & h_4 \\ 0 & 0 & 0 & -1 & h_5 \end{pmatrix}$$

where h is the polynomial defined in (29) and $h_i = \partial h / \partial u_i$, i.e. the matrix is the matrix of the partial derivatives of the polynomials defining Λ with respect to u_1, \dots, u_5 . Equation (31) can be shown by some simple but tedious calculations or by implicit function arguments.

With the help of Macaulay 2 we get:

Proposition 16. *The focal locus X of B_{MA} has the same invariants as the sextic threefold of Example 1: its Hilbert polynomial is $P_X(t) = t^3 + 3t^2 + 2$ and its singular locus is a twisted cubic. Its ideal sheaf has the same resolution as in (20), and by the same reasons it is LCM.*

Remark 7. For particular choices of the quadric in (27) the threefold X can degenerate: for instance, if Q is a quadric such that $a_2 = \dots a_6 = b_1 = \dots b_5 = c = 0$, then X is a scheme with support a cone over a twisted cubic.

Proposition 17. *The lines of the congruence B passing through a general focal point $P \in X$ form a planar pencil.*

Proof. It follows immediately from Lemma 12, observing that, since P is general, it is not contained in L , and that, since it is focal, it belongs to infinitely many lines of B . \square

Corollary 18. *The system of conservation laws defined by (29) and (30) is a non linearizable T -system, i.e. the corresponding congruence of lines is not contained in a linear congruence.*

Proof. It has been shown, for example in [AF01], Section 2, that a T -system is characterized by the fact that each focus of the corresponding congruence B is a fundamental point, and moreover that the lines of B , passing through a general focal point, form a planar pencil. In our case, in view of Proposition 17, it is enough to note that the congruence is not contained in any linear congruence, because it has multidegree $(1, 3, 3)$ by Theorem 15, and is irreducible. \square

We conclude by observing some interesting geometric properties of B . As a subvariety of $\mathbb{G}(1, 5)$ it is embedded in \mathbb{P}^{14} and it results to be a 4-dimensional variety covered by lines, i.e. the planar pencils of Proposition 17.

Theorem 19. *Let B be an irreducible congruence of lines in \mathbb{P}^5 of multidegree $(1, 3, 3)$ constructed as in Theorem 14 for a general choice of the quadric Q .*

- (1) *Through a general point of B there pass 4 lines contained in B .*
- (2) *B is a smooth Fano fourfold in \mathbb{P}^{11} of index 2, of degree 16 and sectional genus 9.*

Proof. The first assertion follows from the fact that on a general line of B there are 4 foci (see [DP05], Proposition 2.1), and through each of these foci there is a planar pencil of lines of B . As for the second assertion, we have performed the proof with the help of Macaulay 2 on a general example. We have computed the equations and the Hilbert polynomial for B , obtaining in particular that its sectional genus is 9. So the general curve section of B is a curve in \mathbb{P}^8 of degree 16 and genus 9 hence is a canonical curve, and B results to be a Fano fourfold in \mathbb{P}^{11} of index 2.

For this general choice we have checked that B is smooth. \square

It would be desirable to have a more theoretic proof of the preceding theorem. Nevertheless, we think that this result is interesting because it gives another interpretation of the Fano fourfold of index two and genus 9. This Fano fourfold B belongs to the list of Mukai, see for example [Mu88], who gives an embedding in $\mathbb{G}(2, 5)$, the Grassmannian of planes in \mathbb{P}^5 . Our result shows that on B there is a rank two vector bundle giving an embedding in $\mathbb{G}(1, 5)$. We plan to return to this argument in a subsequent paper.

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